

# Incompatibility of quantum testers

Michal Sedlák,<sup>1,2</sup> Daniel Reitzner,<sup>2</sup> Giulio Chiribella,<sup>3</sup> and Mário Ziman<sup>2,4</sup>

<sup>1</sup>*Department of Optics, Palacký University, 17. listopadu 1192/12, CZ-771 46 Olomouc, Czech Republic*

<sup>2</sup>*RCQI, Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 84511 Bratislava, Slovakia*

<sup>3</sup>*Department of Computer Science, The University of Hong Kong, Pokfulam Road, Hong Kong*

<sup>4</sup>*Faculty of Informatics, Masaryk University, Botanická 68a, 60200 Brno, Czech Republic*

The existence of incompatible measurement setups is one of the distinctive features of quantum theory. Here we extend the notion of incompatibility from measurements that test preparations to measurements that test dynamical processes. Such measurements, known as testers, consist of the preparation of an input state, the application of the tested process, and a measurement on the output. Mathematically, testers are described by a suitable generalization of the notion of positive operator-valued measure (POVM). Unlike in the case of POVMs, however, testers that commute are not necessarily compatible, and testers that are compatible do not necessarily commute. Given a set of testers, we define the robustness of incompatibility as the minimum amount of mixing needed to make the testers compatible. We show that (i) the robustness is lower bounded by the distinguishability of the input states prepared by the different testers, and (ii) maximum robustness is attained when the input states are orthogonal. An extensive analysis of incompatibility is provided for factorized qubit testers with two outcomes. All our results can be extended to measurements that test quantum processes consisting of multiple time steps.

PACS numbers: 03.65.Ta, 03.67.-a, 03.65.-w, 03.65.Aa

## I. INTRODUCTION

The existence of incompatible measurements is one of the most fundamental traits of quantum theory [1–4]. Incompatibility is rooted in all characteristic features of quantum theory, underlying the phenomena of quantum interference [5], quantum uncertainty [6], quantum non-locality [7, 8] and many others. Quantum incompatibility has important consequences for quantum information processing, such as no-cloning [9] and no information without disturbance [10], and represents a resource for Bell inequality violations or Einstein-Podolsky-Rosen steering [11].

Quantum incompatibility has been extensively studied in standard setting where a system is initially prepared in a given state and undergoes a quantum measurement [12–19]. The measurement’s goal is to test a property of the system’s preparation [20, 21]. However, one can consider more general settings, where the goal of the measurement is to test a property of a dynamical process. Operationally, a test on a process is carried out by (i) preparing the input of the process in a known state (possibly on an extended system), (ii) letting the state evolve through the process, and (iii) performing a measurement on the output. A device that performs the procedure (i)-(iii) will be called a *tester* [22].

To introduce the notion of compatibility for testers, it is useful to consider an example. Imagine that we are given a black box, which transforms the polarization of photons, and that our goal is to find out some of its properties. One property is how the black box preserves the horizontal polarization: to test this property, we have to prepare a horizontally polarized photon and to perform a polarization measurement on the output. The probability that the measurement yields horizontal polarization

is a measure of how well the black box preserves the horizontal polarization. However, this experiment gives no information on the preservation of the vertical polarization. To test how the vertical polarization is preserved, one would have to perform a similar experiment, preparing a vertically polarized photon as input and performing a polarization measurement on the output. Now, a natural question is: are these two experiments compatible? In other words, is there an experiment in which the preservation of both the horizontal and the vertical polarizations is investigated at the same time? Because of the orthogonality of the considered polarizations, it is natural to expect that both these features are compatible. In this particular case, the measurements of the output states are compatible. However, it is difficult to imagine how one could simultaneously prepare the input photon both horizontally and vertically polarized. Intuitively, taking a superposition of horizontal and vertical polarization does not work in this case, because the action of the black box on the superposition does not give enough information about the action of the channel on the individual states that are superposed. Later in the paper we will see that this intuition is correct and that the two testers described above are indeed incompatible.

Intuitively, the incompatibility of testers stems from the “incompatibility” of the states sent to the input and/or from the incompatibility of the quantum measurements performed on the output. In the example discussed above, the incompatibility of test states is the reason for incompatibility of process measurements (the two testers). However, is the incompatibility of either of the constituents sufficient for the phenomenon of incompatibility in the sense of process measurements? In this paper we will provide examples answering this question.

The paper is structured as follows: In Section II we

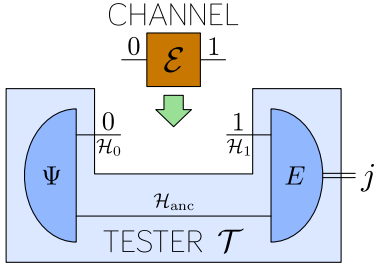


FIG. 1: Pictorial representation of a quantum tester  $\mathcal{T}$  describing some measurement of a quantum process  $\mathcal{E}$ .

will introduce the mathematical framework of process POVMs suited for the analysis of tester incompatibility questions. The Section III introduces a measure of incompatibility of testers that is evaluated in Section IV in cases when the incompatibility is rooted in the incompatibility of final measurements. In Section V we will investigate in details the incompatibility of two-outcome testers, especially, we focus on factorized qubit case. In Section VI we will generalize the incompatibility consideration for general quantum networks and we will point out that the introduced incompatibility measure is bounded from below by success probability characterizing the minimum-error discrimination of corresponding quantum devices.

## II. COMPATIBILITY OF QUANTUM TESTERS

Schematically, any experiment acquiring information on quantum processes can be illustrated as in Fig. 1. In order to distinguish between measurements of states and of processes we will reserve the names *observables* and *testers*, respectively. The tester consists of the preparation of a test state  $\Psi$  (including ancilla) and a measurement  $E$  of the output (i.e. an observable, mathematically formalized as positive operator valued measure, also denoted as POVM). In particular,  $\Psi$  is a density operator defined on  $\mathcal{H}_0 \otimes \mathcal{H}_{\text{anc}}$  and observable  $E$  is described by positive operators  $E_1, \dots, E_n$  (identifying individual outcomes of the observable) such that  $\sum_j E_j = I_1 \otimes I_{\text{anc}}$ , thus, forming a POVM.

The unknown process (input/output device) is filling the gap on the top line, hence, transforming states of a system associated with Hilbert space  $\mathcal{H}_0$  into states of the outgoing system associated with  $\mathcal{H}_1$ . Deterministic quantum processes (*quantum channels*) are identified with completely positive trace preserving linear maps and they can be represented by positive-semidefinite operators  $\Omega \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$  (with  $\text{tr}_1[\Omega] = I_0$ ) also known as Choi-Jamiolkowski operators. For any process  $\mathcal{E}$  we have  $\Omega_{\mathcal{E}} = (\mathcal{E} \otimes \mathcal{I})(\Psi_+)$ , where  $\Psi_+ = \sum_{k,l} |k \otimes k\rangle \langle l \otimes l|$  and  $\{|k\rangle\}$  form an orthonormal basis of  $\mathcal{H}_0$ . For example, the identity channel is represented as  $\Omega_{\mathcal{I}} = \Psi_+$  and

completely depolarizing channel as  $\Omega_0 = \frac{1}{d} I_1 \otimes I_0$ . Let us note that all the results obtained in this paper hold also for completely positive trace non-increasing linear maps (*quantum operations*) describing the probabilistic processes (state transformations) associated with individual outcomes of quantum measurement (see for instance [23]).

Quantum tester  $\mathcal{T}$  (precisely 1-tester [24], or process POVM [25]) is understood as a mapping from processes to probability distributions, i.e.

$$\mathcal{T} : \mathcal{E} \mapsto p_j(\mathcal{E}, \{\Psi, E\}) := \text{tr}[(\mathcal{E} \otimes \mathcal{I}_{\text{anc}})(\Psi)E_j].$$

Define (see for instance [25]) a completely positive linear map  $\mathcal{I}_0 \otimes \mathcal{R}_{\Psi}$  transforming  $\Psi_+$  into  $\Psi$ , where  $\mathcal{R}_{\Psi} : \mathcal{L}(\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H}_{\text{anc}})$ . Using the dual of this map the probability can be rewritten as

$$p_j(\mathcal{E}, \{\Psi, E\}) = \text{tr}[(\mathcal{E} \otimes \mathcal{I}_0)(\Psi_+)(\mathcal{I}_1 \otimes \mathcal{R}_{\Psi}^*)(E_j)],$$

where we recognize the Choi-Jamiolkowski operator and define  $T_j = [(\mathcal{I} \otimes \mathcal{R}_{\Psi}^*)(E_j)]^T$ , where  $T$  denotes transposition in the same basis as was used in the definition of the Choi-Jamiolkowski operator. These positive operators defined on  $\mathcal{H}_1 \otimes \mathcal{H}_0$  contain complete information about the tester's statistics. In particular, the probability of recording an outcome  $j$  when the channel  $\mathcal{E}$  is tested by quantum tester  $\mathcal{T}$  is given by formula

$$p_j(\mathcal{E}, \mathcal{T}) = \text{tr}[T_j \Omega_{\mathcal{E}}^T].$$

As a result the quantum tester  $\mathcal{T}$  is formalized as a specific subnormalized positive operator-valued measure on  $\mathcal{H}_1 \otimes \mathcal{H}_0$ , i.e. a collection of positive operators  $T_j \geq 0$  such that  $\sum_j T_j = I_1 \otimes \varrho$  for some density operator  $\varrho \in \mathcal{S}(\mathcal{H}_0) \equiv \{X \in \mathcal{L}(\mathcal{H}_0) : X \geq 0, \text{tr} X = 1\}$ . For example, the ancilla-free testers composed of a fixed probe state  $\varrho \in \mathcal{S}(\mathcal{H}_0)$  and POVM  $\{E_j\} \in \mathcal{L}(\mathcal{H}_1)$  are described by factorized tester elements  $T_j = E_j^T \otimes \varrho$ . In particular, the experiments described in the introduction example form two-outcome testers

$$\begin{aligned} \mathcal{T}_H : \quad H_+ &= |H\rangle\langle H| \otimes |H\rangle\langle H|, \quad H_- = |V\rangle\langle V| \otimes |H\rangle\langle H|; \\ \mathcal{T}_V : \quad V_+ &= |V\rangle\langle V| \otimes |V\rangle\langle V|, \quad V_- = |H\rangle\langle H| \otimes |V\rangle\langle V|. \end{aligned}$$

Compatibility of quantum testers is defined in the analogous way as for quantum observables, by requirement on the existence of a common device performing both testers.

**Definition 1.** Quantum testers  $\mathcal{A} = \{A_j\}_{j=1}^N$  and  $\mathcal{B} = \{B_k\}_{k=1}^M$  are compatible if there exists a tester  $\mathcal{G} = \{G_{jk}\}$  (with  $NM$  outcomes, thus, indexed by two indexes), such that

$$\sum_k G_{jk} = A_j, \quad \sum_j G_{jk} = B_k. \quad (1)$$

It is a straightforward consequence of the definition that compatible testers have the same normalization.

**Proposition 1.** Suppose testers  $\mathcal{A} = \{A_j\}_{j=1}^N$  and  $\mathcal{B} = \{B_k\}_{k=1}^M$  are compatible. Then  $\sum_j A_j = \sum_k B_k$ .

*Proof.* A direct calculation

$$\begin{aligned} I \otimes \varrho &= \sum_j A_j = \sum_j \sum_k G_{jk} = \sum_k \sum_j G_{jk} = \sum_k B_k \\ &= I \otimes \xi, \end{aligned} \quad (2)$$

implies that  $\varrho \equiv \xi$ , thus, the proposition holds.  $\square$

In the case the normalizations are the same ( $\varrho = \xi$ ) we say they are *compatible*, hence, Proposition 1 says that incompatible normalizations imply incompatibility of the testers. Such observation can be used to argue that testers  $\mathcal{T}_H$  and  $\mathcal{T}_V$  are incompatible, because their normalizations differ, i.e.  $H_+ + H_- \neq V_+ + V_-$ . Moreover, let us stress that all these operators  $H_\pm, V_\pm$  are mutually commuting, hence, we come to a rather surprising conclusion for tester's incompatibility that

$$\text{commutativity} \not\Rightarrow \text{compatibility}.$$

In the following theorem we shall identify additional constrain that should be added to the normalization condition in order to guarantee the compatibility of the testers. For that end, let us denote by  $\mathcal{H}_\varrho$  the subspace of  $\mathcal{H}_0$  determined by the support of  $\varrho$ . For any tester  $\mathcal{A} = \{A_j\}$  normalized to  $I \otimes \varrho$  we introduce a *canonical observable*  $E^{\mathcal{A}}$  composed of positive operators  $E_j^{\mathcal{A}} \equiv (I \otimes \varrho^{-\frac{1}{2}}) A_j (I \otimes \varrho^{-\frac{1}{2}})$  defined on  $\mathcal{H}_1 \otimes \mathcal{H}_\varrho$  and satisfying  $\sum_j E_j^{\mathcal{A}} = I \otimes I_\varrho$ . Let us remind there are many realizations (by means of pairs  $\{\Psi, E\}$ ) for given tester, however, there is always a canonical one in which the bipartite pure state  $|\psi_\varrho\rangle = \sum_l \sqrt{\varrho}|l\rangle \otimes |l\rangle$  is prepared, first sub system is send through the tested quantum channel and both subsystems are measured by the canonical observable. The following necessary and sufficient condition for compatibility employs such canonical POVMs.

**Theorem 1.** Quantum testers  $\mathcal{A} = \{A_j\}_{j=1}^N$  and  $\mathcal{B} = \{B_k\}_{k=1}^M$  are compatible if and only if they have compatible normalizations ( $\sum_k B_k = \sum_j A_j = I \otimes \varrho$ ) and the canonical observables  $E^{\mathcal{A}}$  and  $E^{\mathcal{B}}$  are compatible.

*Proof.* Suppose  $\mathcal{A}, \mathcal{B}$  are compatible. Then there exists a tester  $\mathcal{G}$  such that Eqs.(1) and (2) hold. Define a POVM  $E_{jk}^{\mathcal{G}} = (I \otimes \varrho^{-\frac{1}{2}}) G_{jk} (I \otimes \varrho^{-\frac{1}{2}})$  on  $\mathcal{H}_1 \otimes \mathcal{H}_\varrho$ . Since  $G_{jk} \geq 0$  the operators  $E_{jk}^{\mathcal{G}}$  are positive-semidefinite and the normalization  $\sum_{jk} E_{jk}^{\mathcal{G}} = (I \otimes \varrho^{-\frac{1}{2}})(I \otimes \varrho)(I \otimes \varrho^{-\frac{1}{2}}) = I \otimes I_\varrho$  holds. Clearly,

$$\begin{aligned} \sum_k E_{jk}^{\mathcal{G}} &= (I \otimes \varrho^{-\frac{1}{2}}) \left( \sum_k G_{jk} \right) (I \otimes \varrho^{-\frac{1}{2}}) \\ &= (I \otimes \varrho^{-\frac{1}{2}}) A_j (I \otimes \varrho^{-\frac{1}{2}}) = E_j^{\mathcal{A}} \end{aligned}$$

leads to the desired POVM  $E^{\mathcal{A}}$ . Similarly, we find  $\sum_j E_{jk}^{\mathcal{G}} = E_k^{\mathcal{B}}$ , thus, the compatibility of POVMs  $\{E_j^{\mathcal{A}}\}, \{E_k^{\mathcal{B}}\}$  is shown.

For the opposite implication we suppose POVMs  $\{E_j^{\mathcal{A}}\}, \{E_k^{\mathcal{B}}\}$  are compatible and both testers are normalized to  $I \otimes \varrho$ . Then there exists POVM  $E_{jk}$  such that  $\sum_k E_{jk} = E_j^{\mathcal{A}}, \sum_j E_{jk} = E_k^{\mathcal{B}}$ . Further, let us define positive operators  $G_{jk} = (I \otimes \varrho^{\frac{1}{2}}) E_{jk} (I \otimes \varrho^{\frac{1}{2}})$  with  $\sum_{jk} G_{jk} = I \otimes \varrho$ . This demonstrates that  $\mathcal{G} = \{G_{jk}\}$  is a valid tester. Moreover,

$$\begin{aligned} \sum_k G_{jk} &= (I \otimes \varrho^{\frac{1}{2}}) \left( \sum_k E_{jk} \right) (I \otimes \varrho^{\frac{1}{2}}) \\ &= (I \otimes \varrho^{\frac{1}{2}}) E_j^{\mathcal{A}} (I \otimes \varrho^{\frac{1}{2}}) = A_j \end{aligned}$$

and similarly we obtain  $\sum_j G_{jk} = B_k$ . Thus, the testers  $\mathcal{A}$  and  $\mathcal{B}$  are compatible, which concludes the proof.  $\square$

This theorem can be rephrased in the following way. The compatibility of testers requires the existence of a joint tester implementable as preparation of pure bipartite state  $|\psi_\varrho\rangle$  followed by the measurement of a bipartite POVM  $\{E_{jk}\}$  with marginals being the canonical observables  $\{E_j^{\mathcal{A}}\}$  and  $\{E_k^{\mathcal{B}}\}$ .

The simplest case of a tester in number of outcomes is a two-outcome tester. Incompatibility of this case of two-outcome testers will be considered specifically in later parts of the paper. For these testers we may formulate the following proposition simplifying previous conditions.

**Proposition 2.** Two-outcome quantum testers  $\mathcal{A}$  and  $\mathcal{B}$  with common normalization  $I \otimes \varrho$  are compatible if and only if there exists operator  $G \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$  such that

$$0 \leq G \leq A_1, B_1, \quad (3)$$

$$A_1 + B_1 \leq G + I \otimes \varrho. \quad (4)$$

*Proof.* Setting  $G := G_{11}$  gives  $G_{12}, G_{21}$  and  $G_{22}$  uniquely determined by Eq. (1) and the normalization (2) as

$$\begin{aligned} G_{12} &= A_1 - G \\ G_{21} &= B_1 - G \\ G_{22} &= I \otimes \varrho + G - A_1 - B_1. \end{aligned}$$

The positivity of  $\{G_{jk}\}$  is equivalent to Eqs. (3) and (4), which concludes the proof  $\square$

### III. ROBUSTNESS OF INCOMPATIBILITY

The requirement that the two testers have to have the same normalization to be compatible is highly restrictive. One can learn more about their compatibility by a finer approach that quantifies the incompatibility of testers. We will adopt the procedure that is often used in the case of observables [11, 26]. In particular, consider testers  $\mathcal{A} = \{A_j\}$  and  $\tilde{\mathcal{A}} = \{\tilde{A}_j\}$  with normalizations  $I \otimes \varrho$  and  $I \otimes \tilde{\varrho}$ , respectively. Without loss of generality we will assume that both these testers have the same number of outcomes, however, we fix the labeling of the outcomes. Mixing  $\mathcal{A}$  with  $\tilde{\mathcal{A}}$  mathematically results in

convex combination defining a tester  $\bar{\mathcal{A}}_\lambda = (1-\lambda)\mathcal{A} + \lambda\tilde{\mathcal{A}}$ , where  $0 \leq \lambda \leq 1$  quantifies the amount of  $\tilde{\mathcal{A}}$ -noise added to  $\mathcal{A}$ . Let us stress that  $\bar{\mathcal{A}}_\lambda$  is represented by operators  $\bar{A}_j = (1-\lambda)A_j + \lambda\tilde{A}_j$  and the normalization changes to  $\sum_j \bar{A}_j = I \otimes \bar{\varrho}$  with  $\bar{\varrho} = (1-\lambda)\varrho + \lambda\tilde{\varrho}$ .

Further, let us assume we have a pair of incompatible testers  $\mathcal{A}$  and  $\mathcal{B}$ . We will exploit the described procedure of mixing each of them with different testers  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{B}}$  (with normalizations  $I \otimes \tilde{\varrho}$  and  $I \otimes \tilde{\xi}$ ), respectively.

**Definition 2.** Testers  $\mathcal{A}$  and  $\mathcal{B}$  are said to be  $\lambda$ -compatible ( $\lambda \in [0, 1]$ ) if there exist testers  $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$  such that  $\bar{\mathcal{A}} \equiv (1-\lambda)\mathcal{A} + \lambda\tilde{\mathcal{A}}, \bar{\mathcal{B}} \equiv (1-\lambda)\mathcal{B} + \lambda\tilde{\mathcal{B}}$  are compatible.

**Proposition 3.** An arbitrary pair of testers  $\mathcal{A}, \mathcal{B}$  is  $\lambda$ -compatible with  $\lambda = 1/2$ .

*Proof.* Set  $\tilde{\mathcal{A}} = \mathcal{B}, \tilde{\mathcal{B}} = \mathcal{A}$ . Then we have  $\frac{1}{2}\mathcal{A} + \frac{1}{2}\tilde{\mathcal{A}} = \frac{1}{2}\mathcal{A} + \frac{1}{2}\mathcal{B} = \frac{1}{2}\mathcal{B} + \frac{1}{2}\tilde{\mathcal{B}}$ , which concludes the proof, since a tester is always compatible with itself.  $\square$

Clearly, the  $\lambda$ -compatibility implies  $\lambda'$ -compatibility for all  $\lambda' > \lambda$ . Therefore, for the quantification of incompatibility it is better to focus on the minimal value of  $\lambda$ .

**Definition 3.** The robustness of incompatibility  $R(\mathcal{A}, \mathcal{B})$  is the minimal value of  $0 \leq \lambda \leq 1/2$ , for which the testers  $\mathcal{A}$  and  $\mathcal{B}$  are  $\lambda$ -compatible.

It follows that  $0 \leq R(\mathcal{A}, \mathcal{B}) \leq 1/2$ , where the lower bound is achieved (by definition) only for compatible testers. In the case of compatibility of measurements the upper bound is achievable only for infinite-dimensional quantum systems [11, 26]. In what follow we will see that for testers this bound is achievable already for two-dimensional system.

The necessary condition for compatibility formulated in Proposition 1 will lead us to a simple lower bound on  $R(\mathcal{A}, \mathcal{B})$ .

**Proposition 4.** Testers  $\mathcal{A}$  and  $\mathcal{B}$  with normalizations  $I \otimes \varrho = \sum_j A_j, I \otimes \xi = \sum_k B_k$ , respectively, can be  $\lambda$ -compatible only if there exist states  $\tilde{\varrho}, \tilde{\xi} \in \mathcal{S}(\mathcal{H}_0)$  such that

$$(1-\lambda)\varrho + \lambda\tilde{\varrho} = (1-\lambda)\xi + \lambda\tilde{\xi}. \quad (5)$$

*Proof.* The normalization of  $(1-\lambda)\mathcal{A} + \lambda\tilde{\mathcal{A}}, (1-\lambda)\mathcal{B} + \lambda\tilde{\mathcal{B}}$  reads  $(1-\lambda)I \otimes \varrho + \lambda I \otimes \tilde{\varrho}, (1-\lambda)I \otimes \xi + \lambda I \otimes \tilde{\xi}$ , respectively. Since compatible testers must have the same normalization we obtain the claim.  $\square$

Let us denote by  $\lambda_{\min}$  the minimal value of  $\lambda$  (minimal level of added noise) such that the normalizations of  $\bar{\mathcal{A}}$  and  $\bar{\mathcal{B}}$  are compatible.

In the following, we will provide some insight into the geometry of possible choices of  $\tilde{\varrho}$  and  $\tilde{\xi}$  and we will evaluate the value of  $\lambda_{\min}$ . Let us think of the set of states

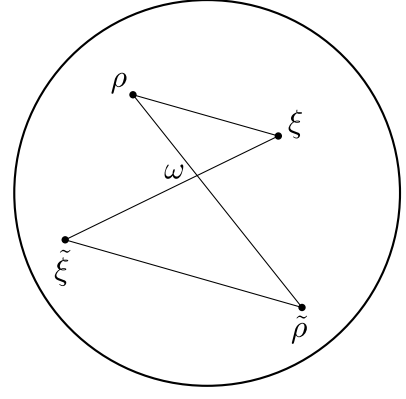


FIG. 2: To obtain common normalization  $I \otimes \omega$  for the same  $\lambda$  lines connecting  $\varrho$  with  $\xi$  and  $\tilde{\varrho}$  with  $\tilde{\xi}$  must be parallel.

$\mathcal{S}(\mathcal{H}_0)$  as a subset of a vector space  $\mathcal{L}(\mathcal{H}_0)$ , which is endowed with a norm  $\|X\| = \text{tr}|X|$  and a norm-induced distance  $d(X, Y) = \|X - Y\| \forall X, Y \in \mathcal{L}(\mathcal{H}_0)$ . Both sides of Eq.(5) determine a segment of line (see Fig. 2). Moreover, this identity guarantees that all points  $\varrho, \xi, \tilde{\varrho}, \tilde{\xi}$  belong to the same plane. The equality represents their intersection and the requirement of the same  $\lambda$  on both sides represents the fact that  $d(\xi, \omega)/d(\tilde{\xi}, \omega) = d(\varrho, \omega)/d(\tilde{\varrho}, \omega)$ , where we set  $\omega = (1-\lambda)\varrho + \lambda\tilde{\varrho} = (1-\lambda)\xi + \lambda\tilde{\xi}$ . It follows from the simple geometry (see Fig. 2) that this is satisfied only if the lines connecting  $\varrho$  with  $\xi$  and  $\tilde{\varrho}$  with  $\tilde{\xi}$  are parallel. Moreover, from Eq. (5) we get  $\lambda = \|\varrho - \xi\|/(\|\varrho - \xi\| + \|\tilde{\varrho} - \tilde{\xi}\|)$ . Clearly, the smaller the  $\lambda$  the longer is the segment of the line determined by  $\tilde{\varrho}$  and  $\tilde{\xi}$ . To determine  $\lambda_{\min}$  we are looking for the longest line contained inside the state space that is parallel to the line determined by states  $\varrho$  and  $\xi$ . The distance between states ( $\tilde{\varrho}$  and  $\tilde{\xi}$ ) is maximal for states with mutually orthogonal supports, i.e. for  $\Pi_{\tilde{\varrho}} \perp \Pi_{\tilde{\xi}}$  the distance is  $\|\tilde{\varrho} - \tilde{\xi}\| = 2$ . Such two states always exist as we shall show, and so using the above considerations we can formulate a lower bound on the robustness of incompatibility of the testers.

**Proposition 5.** For any pair of testers  $\mathcal{A}$  and  $\mathcal{B}$

$$R(\mathcal{A}, \mathcal{B}) \geq \lambda_{\min} = \frac{\|\varrho - \xi\|}{\|\varrho - \xi\| + 2}. \quad (6)$$

*Proof.* First, let us show that for any states  $\varrho$  and  $\xi$  states  $\tilde{\varrho}$  and  $\tilde{\xi}$  exists such that Eq. (5) is satisfied for  $\lambda = \lambda_{\min}$  and  $\|\tilde{\varrho} - \tilde{\xi}\| = 2$ . Let

$$\Delta = \frac{\varrho - \xi}{\|\varrho - \xi\|}.$$

This operator is self-adjoint, traceless and  $\|\Delta\| = 1$ . This implies that we can split operator  $\Delta = \Delta_+ - \Delta_-$  into its positive and negative part ( $\Delta_+, \Delta_- \geq 0$ ) and

$$\begin{aligned} 1 &= \text{tr}\Delta_+ + \text{tr}\Delta_-, \\ 0 &= \text{tr}\Delta_+ - \text{tr}\Delta_-, \end{aligned}$$

This means that  $\text{tr}\Delta_+ = \text{tr}\Delta_- = 1/2$ , which allow us to define states

$$\tilde{\varrho} = 2\Delta_-, \quad \tilde{\xi} = 2\Delta_+.$$

for which we have  $\|\tilde{\varrho} - \tilde{\xi}\| = 2\|\Delta\| = 2$ . Eq. (5) can be for  $\lambda = \lambda_{\min}$  rewritten as  $\frac{1-\lambda_{\min}}{\lambda_{\min}}(\varrho - \xi) = \tilde{\xi} - \tilde{\varrho}$ . It is easy to show that both left and right hand side of the equation equal to  $2\Delta$ .

The lower bound in Eq. (6) is a direct consequence of Proposition 4 and definition of  $R(\mathcal{A}, \mathcal{B})$ . If  $R(\mathcal{A}, \mathcal{B})$  was less than  $\lambda_{\min}$ , then either the Proposition 4 can not be satisfied or  $\lambda_{\min}$  is not the minimal  $\lambda$  achieving compatible normalizations.  $\square$

In the special case of two-dimensional systems (qubits) the geometry of the testers normalizations problem can be easily understood due to the simplicity of Bloch representation (depicted as a sphere on Fig. 2). Having situation as in Proposition 4, the states  $\varrho$  and  $\xi$  correspond to two points within the Bloch ball, i.e.  $\varrho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$  and  $\xi = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma})$ , where  $\vec{\sigma}$  is a vector of Pauli matrices and  $\vec{r}, \vec{s}$  are the so-called Bloch vectors ( $\|\vec{r}\|, \|\vec{s}\| \leq 1$ ) and  $\|\cdot\|$  denotes the Euclidean norm.

**Proposition 6.** *Consider a pair of two-dimensional (qubit) testers  $\mathcal{A}$  and  $\mathcal{B}$  with normalizations  $\varrho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$  and  $\xi = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma})$ , respectively. Then*

$$\lambda_{\min} = \frac{\|\vec{r} - \vec{s}\|}{\|\vec{r} - \vec{s}\| + 2}. \quad (7)$$

*Proof.* It follows from the general consideration before that we are looking for  $\tilde{\varrho}$  and  $\tilde{\xi}$  with orthogonal supports. However, in the case of qubit there is only one possibility. The states  $\tilde{\varrho}$  and  $\tilde{\xi}$  are pure and form an orthonormal basis of the underlying two-dimensional Hilbert space. In particular, the vector  $\vec{r} - \vec{s}$  determines all the lines parallel to the segment of the line determined by  $\varrho$  and  $\xi$ . It follows that orthonormal basis in this directions is composed of pure states  $\tilde{\varrho} = \frac{1}{2}(I - \vec{x} \cdot \vec{\sigma})$  and  $\tilde{\xi} = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma})$ , where  $\vec{x} = (\vec{r} - \vec{s})/\|\vec{r} - \vec{s}\|$  is a unit vector. For such choice of  $\tilde{\varrho}$  and  $\tilde{\xi}$  the identity in Eq. (5) gives the minimal value of  $\lambda_{\min} = (\|\vec{r} - \vec{s}\|)/(\|\vec{r} - \vec{s}\| + 2)$ .  $\square$

#### A. Orthogonal testers

We say that the testers  $\mathcal{A} = \{A_j\}$ , normalized to  $I \otimes \varrho$ , and  $\mathcal{B} = \{B_k\}$ , normalized to  $I \otimes \xi$ , are mutually *orthogonal* ( $\mathcal{A} \perp \mathcal{B}$ ) if their normalizations are orthogonal, i.e.  $\text{tr}(\varrho\xi) = 0$ . Intuitively such pairs represent testers with the strongest normalization's incompatibility. Denote by  $\Pi_\varrho$  projector onto the support of  $\varrho$  and similarly by  $\Pi_\xi$  projector onto the support of  $\xi$ . Since  $A_j \leq I \otimes \Pi_\varrho$ ,  $B_k \leq I \otimes \Pi_\xi$ , and  $\Pi_\xi \Pi_\varrho = O$  it follows that  $\text{tr}A_j B_k = 0$  for all combinations of  $j$  and  $k$ . Moreover,  $[A_j, B_k] = O$  for all values of  $j$  and  $k$ , hence, orthogonality of testers

implies their commutativity, i.e.  $[\mathcal{A}, \mathcal{B}] = O$ . The following theorem relates the concept of orthogonality with incompatibility.

**Theorem 2.** *Orthogonal testers are maximally incompatible, i.e.  $R(\mathcal{A}, \mathcal{B}) = 1/2$  whenever  $\mathcal{A} \perp \mathcal{B}$ .*

*Proof.* Multiplying both sides of Eq. (5) with  $\Pi_\varrho$  and taking trace of the equation we obtain  $(1 - \lambda) + \lambda \text{tr}(\tilde{\varrho} \Pi_\varrho) = \lambda \text{tr}(\tilde{\xi} \Pi_\varrho)$ , where we used  $\text{tr}(\xi \Pi_\varrho) = 0$ . This implies  $\lambda = 1/(1 - \text{tr}(\tilde{\varrho} \Pi_\varrho) + \text{tr}(\tilde{\xi} \Pi_\varrho)) \geq 1/2$ , because  $\text{tr}(\tilde{\varrho} \Pi_\varrho) \geq 0$  and  $\text{tr}(\tilde{\xi} \Pi_\varrho) \leq 1$ .  $\square$

Considered example of polarization testers  $\mathcal{T}_H$  and  $\mathcal{T}_V$  (being experiments on two-dimensional quantum system) provides such a case of orthogonal pair of testers. Consequently, they are maximally incompatible. Let us make one more interesting remark. Not only the commutativity of testers does not imply compatibility, but there are commuting pairs of testers (orthogonal ones) that are maximally incompatible.

#### IV. INCOMPATIBILITY INDUCED BY OBSERVABLES

In the previous section we have seen that the incompatibility of testers induced by their normalizations can be even maximal (for orthogonal testers). Now we will look at how much the incompatibility of testers is influenced by their (canonical) measurements alone. To that end we will consider pairs of incompatible testers  $\mathcal{A}, \mathcal{B}$  with compatible normalizations, i.e.  $\sum_j A_j = \sum_k B_k = I \otimes \varrho$ . Clearly, Proposition 4 implies that necessarily also the admixed testers (making them  $\lambda$ -compatible)  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{B}}$ , must possess the same normalization, i.e.  $\sum_j \tilde{A}_j = \sum_k \tilde{B}_k = I \otimes \tilde{\varrho}$ . Set  $\omega = (1 - \lambda)\varrho + \lambda\tilde{\varrho}$ . Theorem 1 says that testers are  $\lambda$ -compatible if and only if the canonical POVMs

$$\begin{aligned} \{\bar{E}_j^{\mathcal{A}} \equiv (I \otimes \omega^{-\frac{1}{2}})[(1 - \lambda)A_j + \lambda\tilde{A}_j](I \otimes \omega^{-\frac{1}{2}})\}_{j=1}^N \\ \{\bar{E}_k^{\mathcal{B}} \equiv (I \otimes \omega^{-\frac{1}{2}})[(1 - \lambda)B_k + \lambda\tilde{B}_k](I \otimes \omega^{-\frac{1}{2}})\}_{k=1}^M \end{aligned} \quad (8)$$

are compatible. Intuitively, in this case the incompatibility of testers is rooted purely in the incompatibility of the canonical observables. However, the values of the robustness for testers and robustness of the associated canonical observables are not necessarily the same.

The testers  $\mathcal{A}, \mathcal{B}$  with the same normalization  $I \otimes \varrho$  can be implemented in an experiment, in which the pure state  $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_\varrho$  (with marginal  $\omega$ ) is used to probe the channel and the POVM  $E^{\mathcal{A}}$  or  $E^{\mathcal{B}}$  is used to measure the outgoing systems. Define  $r(\mathcal{A}, \mathcal{B})$  as the minimal value of  $\mu$  for which the observables  $\bar{F}^{\mathcal{A}} = (1 - \mu)E^{\mathcal{A}} + \mu\tilde{F}^{\mathcal{A}}$  and  $\bar{F}^{\mathcal{B}} = (1 - \mu)E^{\mathcal{B}} + \mu\tilde{F}^{\mathcal{B}}$  are compatible for a suitable choice of POVMs  $\tilde{F}^{\mathcal{A}}$  and  $\tilde{F}^{\mathcal{B}}$  defined on the same system  $\mathcal{H}_1 \otimes \mathcal{H}_\varrho$ .

**Proposition 7.** *The robustness of incompatibility  $R(\mathcal{A}, \mathcal{B})$  of testers  $\mathcal{A}$  and  $\mathcal{B}$  with compatible normalizations is upper bounded by the robustness of incompatibility  $r(\mathcal{A}, \mathcal{B})$  of the canonical POVMs, i.e.*

$$R(\mathcal{A}, \mathcal{B}) \leq r(\mathcal{A}, \mathcal{B}).$$

*Proof.* It follows directly from the definitions that POVMs  $\bar{F}^{\mathcal{A}}$  and  $\bar{F}^{\mathcal{B}}$  define compatible testers  $\bar{\mathcal{A}}^F$  and  $\bar{\mathcal{B}}^F$  via relations  $\bar{A}_j^F = (I \otimes \varrho^{\frac{1}{2}}) \bar{F}_j^{\mathcal{A}} (I \otimes \varrho^{\frac{1}{2}})$  and  $\bar{B}_k^F = (I \otimes \varrho^{\frac{1}{2}}) \bar{F}_k^{\mathcal{B}} (I \otimes \varrho^{\frac{1}{2}})$ . In particular, define  $\tilde{A}_j^F = (I \otimes \varrho^{\frac{1}{2}}) \tilde{F}_j^{\mathcal{A}} (I \otimes \varrho^{\frac{1}{2}})$  and  $\tilde{B}_k^F = (I \otimes \varrho^{\frac{1}{2}}) \tilde{F}_k^{\mathcal{B}} (I \otimes \varrho^{\frac{1}{2}})$  as testers associated with admixed POVMs  $\tilde{F}^{\mathcal{A}}$  and  $\tilde{F}^{\mathcal{B}}$ . Then

$$\bar{A}_j^F = (1 - \mu)A_j + \mu\tilde{A}_j^F, \quad \bar{B}_k^F = (1 - \mu)B_k + \mu\tilde{B}_k^F, \quad (9)$$

hence, the testers  $\mathcal{A}$  and  $\mathcal{B}$  are  $\mu$ -compatible. However, in general, the admixed testers does not have to be normalized to  $I \otimes \varrho$  and different normalization can potentially result in  $\lambda$ -compatible testers with  $\lambda \leq \mu$ , hence, the inequality  $R(\mathcal{A}, \mathcal{B}) \leq r(\mathcal{A}, \mathcal{B})$  follows.  $\square$

It is natural to ask whether the incompatibility originating in observables can be as strong as the incompatibility determined by incompatible normalizations, for which we found that the maximal value is achieved for arbitrary pair of orthogonal testers. The following example is suggesting that incompatibility of the associated POVM does not lead to maximally incompatible testers. For observables on  $d$ -dimensional system one would expect that the robustness of incompatibility is maximized for a pair of conjugated sharp observables (in analogy with position and momentum), because they provide the most different information on the state of the system.

Suppose  $\{|e_j\rangle\}_j$  is an orthonormal basis of  $\mathcal{H}_1 \otimes \mathcal{H}_0$  and  $\{|f_k\rangle\}$  is its Quantum Fourier transform, i.e.

$$|f_k\rangle = \frac{1}{d} \sum_j e^{\frac{i2\pi}{d^2}jk} |e_j\rangle. \quad (10)$$

It was shown in [27] that the sharp POVMs

$$E_j^{\mathcal{A}} = |e_j\rangle\langle e_j|, \quad E_k^{\mathcal{B}} = |f_k\rangle\langle f_k|, \quad (11)$$

become compatible for all values of the mixing parameters  $\mu$

$$\mu \geq \frac{1}{2} \left(1 - \frac{1}{d}\right). \quad (12)$$

Further, let us consider processes on  $d$ -dimensional Hilbert space, i.e.  $|\mathcal{H}_1| = |\mathcal{H}_0| = d$ . Define a pair of testers  $\mathcal{A}$  and  $\mathcal{B}$  with

$$A_j = \frac{1}{d} |e_j\rangle\langle e_j|, \quad B_k = \frac{1}{d} |f_k\rangle\langle f_k|, \quad (13)$$

respectively. If the considered pair of conjugate POVMs is really the most incompatible one, then the upper bound from Proposition 7 implies that  $R \leq \frac{1}{2} (1 - \frac{1}{d}) <$

$1/2$  for any finite  $d$ . However, we know that there are pairs of (orthogonal) testers for which  $R = 1/2$ . We may conclude that the incompatibility originating in different normalizations is stronger than the incompatibility originating in the incompatibility of the associated POVMs.

For the special case of joint pure normalization, i.e.  $\varrho_\psi = |\psi\rangle\langle\psi|$ , we may formulate the following result.

**Proposition 8.** *If the testers  $\mathcal{A}$  and  $\mathcal{B}$  with common normalization of  $I \otimes \varrho_\psi$  are  $\lambda$ -compatible, then it is possible to choose  $\tilde{\mathcal{A}}$ ,  $\tilde{\mathcal{B}}$  and the joint tester  $\mathcal{G}$  normalized to  $I \otimes \varrho_\psi$ . The purity of  $\varrho_\psi$  implies that operators  $\tilde{A}_j$ ,  $\tilde{B}_k$ ,  $G_{jk}$  have factorized form  $x \otimes \varrho_\psi$  for some positive operator  $x$  on  $\mathcal{H}_1$ .*

*Proof.* Let  $\mathcal{A} = \{A_j\}$ ,  $\mathcal{B} = \{B_k\}$  with  $\sum_j A_j = \sum_k B_k = I \otimes \varrho_\psi$ . Since  $\varrho_\psi$  is pure, then necessarily  $A_j = a_j \otimes \varrho_\psi$  and  $B_k = b_k \otimes \varrho_\psi$  for some positive operators  $a_j$ ,  $b_k$  on  $\mathcal{H}_1$ . Assume that  $\mathcal{A}$  and  $\mathcal{B}$  are  $\lambda$ -compatible, i.e. there exist testers  $\tilde{\mathcal{A}}$ ,  $\tilde{\mathcal{B}}$  with normalizations  $\sum_j \tilde{A}_j = \sum_k \tilde{B}_k = I \otimes \tilde{\varrho}$  for some state  $\tilde{\varrho}$  and  $\mathcal{G}$  such that

$$\begin{aligned} \sum_k G_{jk} &= (1 - \lambda)A_j + \lambda\tilde{A}_j \\ \sum_j G_{jk} &= (1 - \lambda)B_k + \lambda\tilde{B}_k \\ \sum_{jk} G_{jk} &= I \otimes \omega, \end{aligned}$$

where  $\omega = (1 - \lambda)\varrho + \lambda\tilde{\varrho}$ . Set  $\kappa = 1 - \lambda + \lambda \text{tr}[\tilde{\varrho}\varrho_\psi]$  and define new admixed testers and joint observable:

$$\begin{aligned} \tilde{A}'_j &= \frac{1}{\text{tr}[\tilde{\varrho}\varrho_\psi]} (I \otimes \varrho_\psi) \tilde{A}_j (I \otimes \varrho_\psi), \\ \tilde{B}'_k &= \frac{1}{\text{tr}[\tilde{\varrho}\varrho_\psi]} (I \otimes \varrho_\psi) \tilde{B}_k (I \otimes \varrho_\psi), \\ G'_{jk} &= \kappa^{-1} (I \otimes \varrho_\psi) G_{jk} (I \otimes \varrho_\psi). \end{aligned}$$

All these new operators are positive and of the factorized form  $x \otimes \varrho_\psi$ . Using the fact that  $\kappa^{-1} \varrho_\psi \omega \varrho_\psi = \varrho_\psi$  it follows that

$$\sum_{jk} G'_{jk} = \sum_j \tilde{A}'_j = \sum_k \tilde{B}'_k = I \otimes \varrho_\psi.$$

Furthermore, we find that

$$\begin{aligned} \sum_k G'_{jk} &= (1 - \lambda')A_j + \lambda'\tilde{A}'_j, \\ \sum_j G'_{jk} &= (1 - \lambda')B_k + \lambda'\tilde{B}'_k, \end{aligned}$$

where we have set  $\lambda' = \lambda \kappa^{-1} \text{tr}[\tilde{\varrho}\varrho_\psi]$ . This result shows that if  $\mathcal{A}$  and  $\mathcal{B}$  are  $\lambda$ -compatible then they are also  $\lambda'$ -compatible with the admixed testers and joint tester of the factorized form  $x \otimes \varrho_\psi$ . Since  $\text{tr}[\tilde{\varrho}\varrho_\psi] \leq 1$  it follows that  $\lambda' \leq \lambda$ , which implies that  $\mathcal{A}$  and  $\mathcal{B}$  are  $\lambda$ -compatible with factorized admixtures and joint observable.  $\square$

The above proposition 8 implies that for this class of testers their  $\lambda$ -compatibility reduces to  $\lambda$ -compatibility of the associated POVMs given in Eq. (8).

## V. TWO OUTCOME TESTERS

Let us focus now on the (in)compatibility of two outcome testers  $\mathcal{A} = \{A_1, A_2\}$  and  $\mathcal{B} = \{B_1, B_2\}$ . Even in this simplest case we will see that the situation is more involved than in the case of POVM compatibility. Firstly, it is not sufficient to investigate only  $A_1$  and  $B_1$  as in the case of compatibility of two-outcome POVMs. For testers the freedom in the normalization must be taken into account, as in Proposition 2. In an extreme case, even if  $A_1 = B_1$  it does not mean the testers  $\mathcal{A}$  and  $\mathcal{B}$  are compatible. In fact, different normalizations imply they are not.

This additional constraint induces the need for the robustness as will be shown in the following proposition. While in the case of two-outcome POVMs their compatibility implies their compatibility [28], in similar situation for testers one does not obtain compatible testers, only such for which the robustness of incompatibility is determined by the compatibility of their normalizations.

**Proposition 9.** *Consider a pair of two outcome testers  $\mathcal{A}$  and  $\mathcal{B}$  such that either  $A_j \leq B_k$ , or  $B_k \leq A_j$  for some combination of  $j, k \in \{1, 2\}$ . Then  $R(\mathcal{A}, \mathcal{B}) = \lambda_{\min}$ .*

*Proof.* Let us focus on the case  $A_1 \leq B_1$ . The other cases can be solved in a completely analogous way. Suppose we identified  $\lambda_{\min}$  and states  $\tilde{\rho}, \tilde{\xi}$  such that Eq.(5) holds. Define testers  $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$  and  $\mathcal{G}$  as follows

$$\begin{aligned} \tilde{A}_1 &= 0; & G_{11} &= (1 - \lambda)A_1; \\ \tilde{A}_2 &= I \otimes \tilde{\rho}; & G_{12} &= 0; \\ \tilde{B}_1 &= 0; & G_{21} &= (1 - \lambda)(B_1 - A_1); \\ \tilde{B}_2 &= I \otimes \tilde{\xi}; & G_{22} &= (1 - \lambda)(I \otimes \xi - B_1) + \lambda I \otimes \tilde{\xi}. \end{aligned}$$

Using Eq.(5) it is easy to show that the relations

$$\begin{aligned} \sum_j G_{ij} &= (1 - \lambda)A_i + \lambda \tilde{A}_i, \\ \sum_i G_{ij} &= (1 - \lambda)B_j + \lambda \tilde{B}_j, \end{aligned}$$

are fulfilled. Thus, we demonstrated that  $R(\mathcal{A}, \mathcal{B}) = \lambda_{\min}$ .  $\square$

### A. Case study: Qubit rank-1 factorized testers

We have used polarization example on several occasions already. In this part we will show generalized results. In particular we will focus on experiments in which the qubit processes are probed with pure qubit states (no ancilla is involved) and at the output some sharp qubit observable is performed. In other words, we analyze how the process preserves polarization determined by the choice of the measurement (polarization filter) provided that initially the photon had some specific polarization. In a sense this is the simplest possible case to investigate.

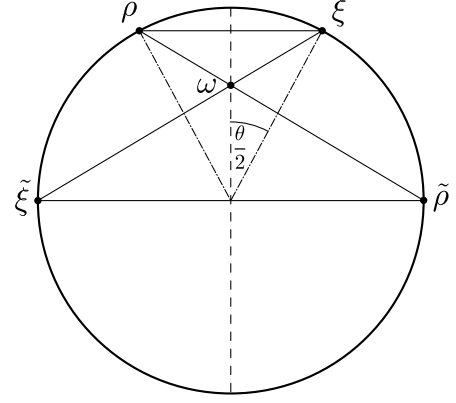


FIG. 3: Illustration of the optimal choice of the normalizations  $\tilde{\rho}, \tilde{\xi}$  of the admixed testers  $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$  for the incompatibility of testers defined in Eq. (14)

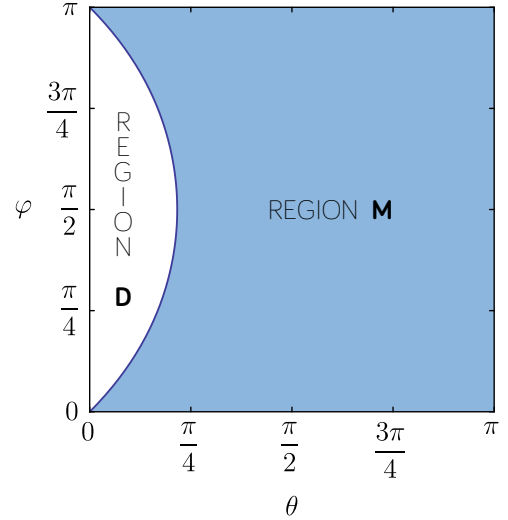


FIG. 4: Region  $M$  defined before Proposition 10 is characteristic by having the robustness of incompatibility  $R(\mathcal{A}, \mathcal{B})$  equal to the lowest possible  $\lambda_{\min}$  making the normalizations compatible.

Mathematically, these experiments are described by two-outcome testers  $\mathcal{A} = \{A_1, A_2\}$ ,  $\mathcal{B} = \{B_1, B_2\}$ ,

$$\begin{aligned} A_1 &= P_{-\varphi/2} \otimes P_{-\theta/2}, & B_1 &= P_{\varphi/2} \otimes P_{\theta/2}, \\ A_2 &= P_{\pi-\varphi/2} \otimes P_{-\theta/2}, & B_2 &= P_{\varphi/2-\pi} \otimes P_{\theta/2}, \end{aligned} \quad (14)$$

where  $P_\alpha = \frac{1}{2}(I + \sin \alpha \sigma_x + \cos \alpha \sigma_z)$ ,  $\theta \in [0, \pi]$ ,  $\varphi \in [0, \pi]$  are proportional to the angles between the photon polarizations ( $\varrho = P_{-\theta/2}$  and  $\xi = P_{\theta/2}$ ) and used polarization filters ( $P_{-\theta/2}$  and  $P_{\theta/2}$ ), respectively.

Equation (7) implies that normalizations can be compatible for any (see also Fig .3)

$$\lambda \geq \lambda_{\min} = \frac{\sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}}, \quad (15)$$

hence we have the lower bound on the value of  $R(\mathcal{A}, \mathcal{B})$ . The following proposition characterizes for which pairs of testers (of the considered type) this lower bound is achievable, i.e.,  $R(\mathcal{A}, \mathcal{B}) = \lambda_{\min}$ . For this purposes we split the set of all pairs of testers (see Fig. 4) into a region

$$M \equiv \left\{ (\theta, \varphi) \mid \theta, \varphi \in [0, \pi], \sin \frac{\theta}{2} \geq \frac{\sin \varphi}{2 + \sin \varphi} \right\}$$

and its complement, a region  $D = M^c$ .

**Proposition 10.** *If a pair of testers  $\mathcal{A}, \mathcal{B}$  defined by Eq. (14) belongs to region  $M$ , then*

$$R(\mathcal{A}, \mathcal{B}) = \frac{\sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}}.$$

*Proof.* We will prove the proposition by demonstrating a particular choice of testers  $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$  and a joint tester  $\mathcal{G}$  satisfying all the requirements of the robustness of incompatibility for  $\lambda = \lambda_{\min}$  as specified in Eq. (15). We set

$$\begin{aligned} \tilde{A}_1 &= \left[ \frac{1-\delta}{2} P_{(\varphi+\pi)/2} + \frac{1+\delta}{2} P_{(\varphi-\pi)/2} \right] \otimes P_{\pi/2}, \\ \tilde{A}_2 &= I \otimes \tilde{\varrho} - \tilde{A}_1, \\ \tilde{B}_1 &= \left[ \frac{1-\delta}{2} P_{-(\varphi+\pi)/2} + \frac{1+\delta}{2} P_{-(\varphi-\pi)/2} \right] \otimes P_{-\pi/2}, \\ \tilde{B}_2 &= I \otimes \tilde{\xi} - \tilde{B}_1, \end{aligned} \quad (16)$$

where

$$\delta = -\frac{\sin \varphi}{2} \frac{1 - \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}}. \quad (17)$$

Let us stress that the associated states  $\tilde{\varrho} = P_{\pi/2}$  and  $\tilde{\xi} = P_{-\pi/2}$  are orthogonal as it is required in order to saturate the bound (15). By definition we can express  $\omega$  as

$$\begin{aligned} \omega &= \frac{1}{2} \left[ (1-\lambda)(\varrho + \xi) + \lambda(\tilde{\varrho} + \tilde{\xi}) \right] \\ &= \frac{1-\lambda}{2} (P_{-\theta/2} + P_{\theta/2}) + \frac{\lambda}{2} I, \end{aligned} \quad (18)$$

which will be convenient in subsequent calculations. We define the joint tester  $\mathcal{G} = \{G_{11}, G_{12}, G_{21}, G_{22}\}$  as follows

$$\begin{aligned} G_{11} &= G & G_{12} &= \bar{A}_1 - G \\ G_{21} &= \bar{B}_1 - G & G_{22} &= I \otimes \Omega + G - \bar{A}_1 - \bar{B}_1, \end{aligned}$$

where

$$\begin{aligned} G &= \frac{1-\lambda}{2} (\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} \sin \frac{\theta}{2}) [ |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| ] \\ &\quad + \frac{1-\lambda}{2} \cos \frac{\varphi}{2} \cos \frac{\theta}{2} [ |v_1\rangle\langle v_2| + |v_2\rangle\langle v_1| ] \end{aligned} \quad (19)$$

and

$$\begin{aligned} |v_1\rangle &= |\frac{\varphi}{2}\rangle |\frac{\pi}{2}\rangle & |v_2\rangle &= |-\frac{\varphi}{2}\rangle |-\frac{\pi}{2}\rangle \\ |\beta\rangle &= \cos \frac{\beta}{2} |0\rangle + \sin \frac{\beta}{2} |1\rangle. \end{aligned}$$

To demonstrate  $\lambda$ -compatibility of  $\mathcal{A}, \mathcal{B}$  it suffices to show (see Proposition 2) that

$$0 \leq G \leq \bar{A}_1, \bar{B}_1 \quad (20)$$

$$\bar{A}_1 + \bar{B}_1 \leq G + I \otimes \omega \quad (21)$$

holds. Since  $\langle v_1 | v_2 \rangle = 0$  the nonzero eigenvalues of  $G$  are the same as for the matrix

$$\frac{1-\lambda}{2} \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad (22)$$

where  $a = \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} \sin \frac{\theta}{2}$ ,  $b = \cos \frac{\varphi}{2} \cos \frac{\theta}{2}$ . After some algebra the requirement of non-negativity of the eigenvalues leads to the definition of the region  $M$ . Thus, in region  $M$  we proved  $G \geq 0$ .

Let us define

$$D \equiv \bar{A}_1 + \bar{B}_1 - I \otimes \omega \quad (23)$$

$$Q \equiv P_{\varphi/2} \otimes P_{\pi/2} + P_{-\varphi/2} \otimes P_{-\pi/2} \quad (24)$$

$$Q^\perp = I - Q \quad (25)$$

$$S \equiv \sigma_X \otimes \sigma_Z, \quad (26)$$

where  $\sigma_X, \sigma_Z$  are the Pauli matrices. Then Eq. (21) can be rewritten as  $D \leq G$ . In the appendix A we prove

$$D = QDQ + Q^\perp DQ^\perp = G - SGS, \quad (27)$$

which implies  $D \leq G$ , because  $G - D = SGS \geq 0$  due to preservation of eigenvalues of  $G \geq 0$  by unitary rotation  $S$ . Thus, we proved Eq. (21).

Next, we show that due to symmetry of the problem

$$G \leq \bar{A}_1 \Leftrightarrow G \leq \bar{B}_1 \quad (28)$$

For this purpose we define hermitian and unitary operator  $T \equiv \sigma_Z \otimes \sigma_Z$ , for which  $T^2 = I$ . It is easy to verify by direct calculation from Eqs. (14), (16), (19) that

$$TGT = G \quad T\bar{A}_1T = \bar{B}_1. \quad (29)$$

Since conjugation with  $T$  is reversible and preserves eigenvalues we get

$$G \leq \bar{A}_1 \Leftrightarrow TGT \leq T\bar{A}_1T \Leftrightarrow G \leq \bar{B}_1, \quad (30)$$

where we used (29).

In the following we prove  $G \leq \bar{A}_1$  by demonstrating positivity of the matrix of  $\bar{A}_1 - G$  in the basis  $\{|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle\}$ , where

$$\begin{aligned} |v_3\rangle &= S|v_2\rangle = |\frac{\varphi}{2} - \pi\rangle |\frac{\pi}{2}\rangle \\ |v_4\rangle &= S|v_1\rangle = |\pi - \frac{\varphi}{2}\rangle |-\frac{\pi}{2}\rangle \end{aligned}$$



Direct calculation of the matrix elements yields

$$\bar{A}_1 - G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & y & 0 \\ 0 & y & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (31)$$

where

$$\begin{aligned} x &= \frac{1}{2} \left( 1 - \frac{\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} \sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}} \right) \\ y &= \frac{\sin \frac{\varphi}{2} \cos \frac{\theta}{2}}{2(1 + \sin \frac{\theta}{2})} \\ z &= \frac{1}{2} \left( 1 - \frac{\sin^2 \frac{\varphi}{2} \sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}} \right). \end{aligned} \quad (32)$$

Thus, it suffice to examine eigenvalues of matrix

$$W = \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \quad (33)$$

which can be analytically shown to be non-negative  $\forall \theta, \varphi \in [0, \pi]$ . In conclusion we proved validity of Eqs. (20), (21) for  $\forall (\theta, \varphi) \in M$  and thus demonstrated the existence of the joint tester  $\mathcal{G}$  needed for proving  $R(\mathcal{A}, \mathcal{B}) = \lambda_{\min}$  claimed in the proposition.  $\square$

In region  $D$  the situation gets more involved. The incompatibility of testers  $\mathcal{A}, \mathcal{B}$  was numerically studied via SDP, as outlined in Appendix C. We were unable to obtain closed forms of mixed-in testers, or the joint tester. We have observed that these are rank one with normalizations of admixed testers being not orthogonal. This implies that the value of  $R(\mathcal{A}, \mathcal{B})$  does not follow Eq. (15) any more — the situation is depicted in Fig. 5.

Let us note that for  $\theta \geq 2 \arcsin(1/3)$  qubit testers (14) are  $\lambda$ -compatible with  $\lambda$  defined by Eq. (15) independently on the choice of  $\varphi$ . This fact can be utilized for judging compatibility of slightly more general testers, which correspond to pure state preparation and an arbitrary qubit measurement at the output of the process.

**Proposition 11.** *Consider a pair of testers  $\mathcal{A}$  and  $\mathcal{B}$  given as*

$$\begin{aligned} A_1 &= E_1 \otimes P_{-\theta/2} & B_1 &= F_1 \otimes P_{\theta/2} \\ A_2 &= E_2 \otimes P_{-\theta/2} & B_2 &= F_2 \otimes P_{\theta/2} \end{aligned} \quad (34)$$

where  $\{E_1, E_2\}, \{F_1, F_2\}$  are arbitrary qubit POVMs. Then for  $\theta \geq 2 \arcsin(1/3) \approx 0.6797$

$$R(\mathcal{A}, \mathcal{B}) = \frac{\sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}}.$$

*Proof.* See Appendix B  $\square$

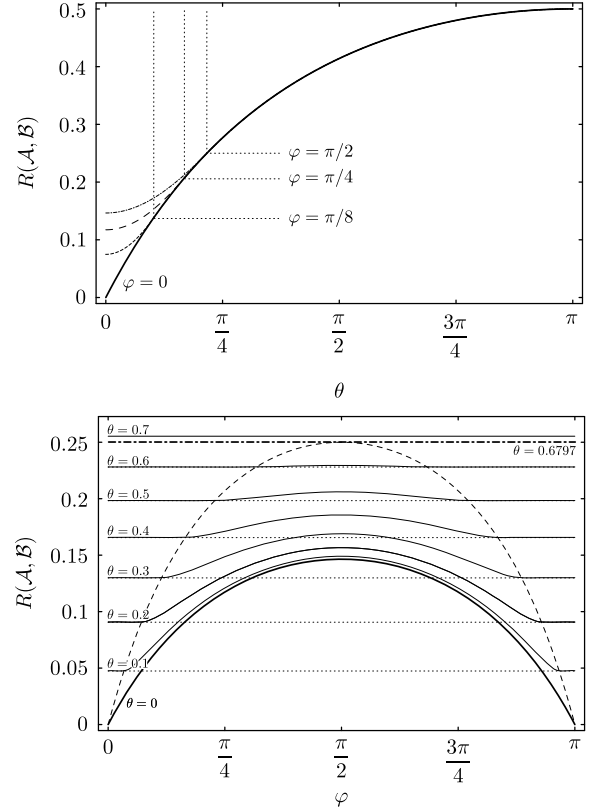


FIG. 5: Robustness of incompatibility of two qubit testers as defined in Eq. (14). Upper figure shows dependence on  $\theta$  and situations for various choices of  $\varphi$ . Solid line depicts bound of Eq. (15). Bottom figure shows dependence on  $\varphi$  for various choices of  $\theta$ . Dotted lines represent Eq. (15) which coincides with  $R(\mathcal{A}, \mathcal{B})$  for  $\theta \geq 0.6797$ .

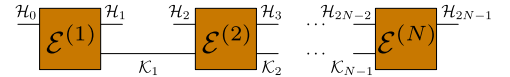


FIG. 6: Illustration of quantum  $N$ -comb  $R^{(N)}$ . Here  $\mathcal{H}_i, i \in \{0, 1, \dots, 2N-1\}$  are the Hilbert spaces describing the inputs and outputs of the network, and  $\mathcal{K}_l, l \in \{1, \dots, N-1\}$  are the Hilbert spaces of the internal memories, used to convey information from one time step to the next.

## VI. EXTENSION TO $N$ -TESTERS

So far we discussed the compatibility of a pair of process measurements designed to test quantum channels, however, the definitions and also most of the results hold in a more general settings. In one direction we may consider (in)compatibility of more than a pair of quantum devices and in another direction we may consider measurements that test a whole causal network, consisting of an ordered sequence of quantum processes correlated by the presence of a quantum memory (see Fig.6).

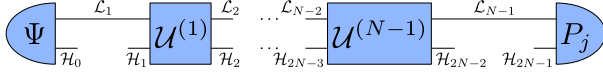


FIG. 7: Illustration of quantum  $N$ -tester resulting in an outcome  $T_j$ . Here  $\mathcal{H}_j \in \{1, \dots, N\}$  are the Hilbert spaces of the internal memories of the network,  $\Psi$  is an input state (which can be chosen to be pure without loss of generality),  $\mathcal{U}^{(l)} \in \{1, \dots, N-1\}$  are quantum channels (which can be chosen to be unitary without loss of generality), and  $\{P_j\}_{j \in J}$  is a POVM, representing a measurement on the final output systems of the network.

We call a sequential quantum network of such form a *quantum  $N$ -comb* [24, 29] (see also Gutoski and Watrous [30]). Two  $N$ -combs are *of the same type* if they have the same sequence of input/output Hilbert spaces. Like quantum channels, quantum  $N$ -combs can be represented by operators. An operator  $R^{(N)}$  is a quantum  $N$ -comb (illustrated in Fig. 6) if and only if it is positive and satisfies the equations

$$\begin{aligned} \text{tr}_{2N-1} [R^{(N)}] &= I_{2N-2} \otimes R^{(N-1)} \\ \text{tr}_{2N-3} [R^{(N-1)}] &= I_{2N-4} \otimes R^{(N-2)} \\ &\vdots \\ \text{tr}_1 [R^{(1)}] &= I_0 \end{aligned} \quad (35)$$

where  $\text{tr}_n$  denotes the partial trace over the Hilbert space  $\mathcal{H}_n$ ,  $I_n$  denotes the identity operator on  $\mathcal{H}_n$  and  $R^{(n)}$  is a positive operator on  $\bigotimes_{i=0}^{2n-1} \mathcal{H}_i$ .

In order to extract information about the quantum network we need to perform an experiment “feeding” all the open legs with particular inputs and outputs. The most general way to interact is by connecting the original network with another sequential network as it is illustrated in Fig. 7.

We call a network of the above form an  *$N$ -tester* [22, 24]. Two  $N$ -testers are *of the same type* if they have the same sequence of input/output Hilbert spaces. An  $N$ -tester illustrated in Fig. 7 is described by a set of positive operators  $\mathbf{T} = \{T_j\}_{j \in J}$  acting on  $\mathcal{H}_{\text{network}} = \bigotimes_{i=0}^{2N-1} \mathcal{H}_i$  satisfying the following set of equations

$$\begin{aligned} \sum_{j \in J} T_j &= I_{2N-1} \otimes \Theta^{(N)} \\ \text{tr}_{2N-2} [\Theta^{(N)}] &= I_{2N-3} \otimes \Theta^{(N-1)} \\ &\vdots \\ \text{tr}_2 [\Theta^{(2)}] &= I_1 \otimes \Theta^{(1)} \\ \text{tr} [\Theta^{(1)}] &= 1, \end{aligned} \quad (36)$$

where  $\Theta^{(n)}$  is a positive operator on  $\bigotimes_{i=0}^{2n-2} \mathcal{H}_i$ . We call

such positive operator  $\Theta^{(N)} \equiv \Theta$  the *normalization* of the tester  $\mathbf{T}$ .

When quantum  $N$ -comb and quantum  $N$ -tester are combined the probability of the outcome  $j \in J$  is given by

$$p_j = \text{tr}[R^{(N)} T_j^T].$$

Different  $N$ -testers represent different (and possibly complementary) ways to extract information about a quantum  $N$ -comb. In the following we will formulate the elementary properties of compatibility for two or more  $N$ -testers. Let us stress that previous sections treat the case of 1-testers, which we for simplicity denoted until now as testers and 1-combs traditionally called channels.

**Definition 4.** Let  $\{\mathbf{T}^{(x)}, x \in X\}$  be a set of testers of the same type, the  $x$ -th tester with outcomes in the set  $J_x$  and the normalization  $\Theta_x$ . The testers are compatible if there exists a joint tester  $\mathbf{G} = \{G_{\vec{j}}\}_{\vec{j} \in J_1 \times \dots \times J_{|X|}}$  such that for all  $x \in X$  and  $j_x \in J_x$

$$T_{j_x}^{(x)} = \sum_{\vec{j}: \vec{j}_x = j_x} G_{\vec{j}}, \quad (37)$$

where  $\vec{j}_x$  stands for the  $x$ -th component of vector  $\vec{j}$ .

As in case of compatibility of pairs of 1-testers, one can show it is necessary that for compatibility of all the considered testers  $\{\mathbf{T}^{(x)}\}_{x \in X}$  their normalizations must coincide. Moreover, the Proposition 1 holds also in this generalized settings.

**Proposition 12.** For each  $N$ -tester  $\mathbf{T}^{(x)}$  with normalization  $\Theta_x$  define a POVM  $\mathbf{P}^{(x)} = \{P_{j_x}^{(x)}\}$

$$P_{j_x}^{(x)} = (I_{2N-1} \otimes \Theta_x^{-\frac{1}{2}}) T_{j_x}^{(x)} (I_{2N-1} \otimes \Theta_x^{-\frac{1}{2}}).$$

The testers  $\{\mathbf{T}^{(x)}, x \in X\}$  are compatible if and only if their normalizations coincide ( $\Theta_x \equiv \Theta$  for all  $x$ ) and the associated observables  $\{\mathbf{P}^{(x)}, x \in X\}$  are compatible.

*Proof.* The proof is a direct generalization of the proof of Proposition 1.  $\square$

The idea of quantification of incompatibility is in perfect analogy with the case of incompatibility for pairs of 1-testers investigated in previous sections. Again, the idea is to measure the incompatibility based on the amount of “noise” that one has to add in order to make the testers compatible.

**Definition 5.** The testers  $\{\mathbf{T}^{(x)}\}_{x \in X}$  are  $\lambda$ -compatible if for any  $x$  there exists a tester  $\tilde{\mathbf{T}}^{(x)}$  with the outcome set  $J_x$  such that the testers  $\{(1-\lambda)\mathbf{T}_x + \lambda\tilde{\mathbf{T}}_x, x \in X\}$  are compatible.

**Definition 6.** The robustness of incompatibility of a set of testers is  $R = \lambda_{\min}$ , the minimum  $\lambda$  such that the testers in the set are  $\lambda$ -compatible.

Note that every set of testers is  $\lambda$ -compatible with  $\lambda = 1 - |X|^{-1}$ , as one can see from a simple adaptation of Proposition 3. In other words  $0 \leq R \leq 1 - |X|^{-1}$  and we will see that the upper bound is achievable. In what follows we will formulate a lower bound in terms of normalization operators  $\Theta_x$ . The crucial observation is that the normalization operator  $\Theta_x$  is essentially the "marginal" of the tester  $T^{(x)}$ , arising when the outcome and the system  $\mathcal{H}_{2N-1}$  are discarded, hence, the operators  $\Theta_x$  represent an  $N$ -comb. We know that a necessary condition for compatibility is that these  $N$ -combs represented by the operators  $\{\Theta_x, x \in X\}$  coincide. This necessary condition leads to a lower bound on the robustness of incompatibility, in which one can recognize success probability for discrimination of testers' normalizations.

**Proposition 13.** *Let  $\{T^{(x)}, x \in X\}$  be a set of testers of the same type with normalizations  $\Theta_x$ , respectively. Then the robustness of incompatibility of this set is lower bounded by*

$$R \geq 1 - \frac{1}{|X| p_{\text{succ}}},$$

where  $p_{\text{succ}}$  is the maximum probability of success in distinguishing among the quantum networks associated with operators  $\{\Theta_x, x \in X\}$ . In particular, when the networks are perfectly distinguishable then the bound is saturated.

*Proof.* Assume the testers are  $\lambda$ -compatible for a certain  $\lambda$  and let  $G = \{G_{\vec{j}}\}_{\vec{j}}$  be the joint tester that guarantees the compatibility. Let us denote by  $\omega^{(N)}$  the normalization of the joint tester. Then necessarily for all  $x \in X$

$$\omega^{(N)} \geq (1 - \lambda) \Theta_x. \quad (38)$$

In order to compute the robustness, we have to minimize  $\lambda$  over all operators  $\omega^{(N)}$  subject to the constraint that  $\omega^{(N)}$  is the normalization of a joint tester satisfying the compatibility condition. We now relax this constraint and assume only that  $\omega^{(N)}$  should be the operator of a quantum network of the desired type. Defining  $\mu = (1 - \lambda)^{-1}|X|^{-1}$ , minimizing  $\lambda$  under the condition (38) is equivalent to minimizing  $\mu$  under the condition

$$\mu \omega^{(N)} \geq \frac{1}{|X|} \Theta_x \quad \forall x \in X. \quad (39)$$

Now, Theorem 1 of Ref. [31] guarantees that the minimum of  $\mu$  under the condition that  $\omega^{(N)}$  is an  $N$ -comb is equal to the maximum probability of success  $p_{\text{succ}}$ . Hence, we must have  $(1 - \lambda) \leq 1/(|X| p_{\text{succ}})$ , which implies  $\lambda \geq 1 - 1/(|X| p_{\text{succ}})$ . Since this relation holds for every  $\lambda$ , it must hold in particular for  $\lambda_{\min}$ , leading to the bound  $R \geq 1 - 1/(|X| p_{\text{succ}})$ . If the quantum networks  $\{\Theta_x, x \in X\}$  are perfectly distinguishable, one has  $p_{\text{succ}} = 1$  and, therefore,  $R \geq 1 - 1/|X|$ . On the other hand, we already mentioned that every set of testers is  $\lambda$ -compatible with  $\lambda = 1 - |X|^{-1}$ , which concludes the proof.  $\square$

In the special case  $|X| = 2$ , i.e. compatibility of a pair of  $N$ -testers, the above bound has a nice expression in terms of the *operational distance* between two quantum networks [22, 24, 32]. Specifically, one has

$$p_{\text{succ}} = \frac{1}{2} \left( 1 + \frac{1}{2} \|\Theta_1 - \Theta_2\|_{\text{op}} \right),$$

where  $\|\cdot\|_{\text{op}}$  is the operational norm [22]. Inserting this expression in the lower bound we then obtain

$$R \geq \frac{\|\Theta_1 - \Theta_2\|_{\text{op}}}{2 + \|\Theta_1 - \Theta_2\|_{\text{op}}}. \quad (40)$$

Let us note that in the case of 1-testers the operational norm coincides with the trace norm, which implies that Eqs. (6) and (40) match.

## VII. CONCLUSIONS

In this paper we introduced the notion of incompatibility of quantum testers and we explored its basic properties. Although there are many common features between the incompatibility of state and process measurements (i.e. observables and testers), there are also fundamental differences. The first difference is that the commutativity of two testers does not imply their compatibility—even if the testers consist of orthogonal operators. This feature is in clear contrast with an elementary feature of incompatibility of quantum measurements (observables), for which it is well known that commutativity implies compatibility.

We have introduced the concept of orthogonality of testers that does not have any analogue in the framework of quantum observables. For observables (unlike for testers) it cannot happen that effects forming different observables are supported on mutually orthogonal subspaces. We have shown that any pair of orthogonal testers are commutative, but also maximally incompatible. In fact, all of them reach the maximal value of robustness,  $R = 1/2$ . And as they exist in any dimension, the maximal incompatibility is not only infinite dimensional phenomenon like in the case of observables [11, 33].

Testers are always implementable as a state preparation on a system extended by an ancilla and subsequent measurement on the extended system. Consider a pair of testers  $\mathcal{A}$  and  $\mathcal{B}$  with implementations  $\{\Psi, A\}$  and  $\{\Phi, B\}$ , hence,  $\Psi, \Phi$  are used as test states and  $A, B$  are the performed observables in implementations of  $\mathcal{A}$  and  $\mathcal{B}$  (see Fig. 1). One could be tempted to determine compatibility of testers by their implementations, but these are not uniquely determined which usually makes such attempts futile. The only case, when the compatibility of testers can be deduced is when the states are the same and the measurements are compatible. On the other hand one can have two different implementations of the same tester with two different input states, which shows, that unequal input states do not imply (in)compatibility

of testers. The most surprising is probably the case, when  $\Psi \neq \Phi$  and  $A \circ B$ , where we used the symbol  $\circ$  to express the incompatibility and we shall use  $\otimes$  to denote the compatibility relation. This case is tempting one to say that the testers are incompatible, yet still even in this case, the two implementations may correspond to the same tester. To summarize,

$$\begin{aligned} \Psi = \Phi \text{ and } A \otimes B &\Rightarrow \mathcal{A} \otimes \mathcal{B}, \\ \Psi \neq \Phi \text{ and } A \otimes B &\not\Rightarrow \mathcal{A} \circ \mathcal{B}, \\ \Psi = \Phi \text{ and } A \circ B &\not\Rightarrow \mathcal{A} \circ \mathcal{B}, \\ \Psi \neq \Phi \text{ and } A \circ B &\not\Rightarrow \mathcal{A} \circ \mathcal{B}. \end{aligned}$$

In the light of previous discussion it is clear that the elements of testers, namely the preparation state and measurement, are not suitable for determining the testers' incompatibility. In fact, we show one should look at the tester's normalizations and induced canonical observables. The incompatibility of normalizations of testers determines a lower bound on tester's incompatibility  $\lambda_{\min} \leq R(\mathcal{A}, \mathcal{B})$ . For the special case of normalization compatible testers, the robustness of tester's incompatibility is upper bounded by the robustness of the associated canonical observables ( $R(\mathcal{A}, \mathcal{B}) \leq r(\mathcal{A}, \mathcal{B})$ ) (see section IV for details).

In conclusion, in order to verify the compatibility of testers it is necessary to verify that normalization states are compatible and also the associated canonical observables are compatible. We have investigated in details the compatibility of factorized qubit two-outcome testers. We found that for a relatively large region (M) the robustness of incompatibility is completely determined by the normalizations of the testers (representing the probe states), however, as it is illustrated in Fig. 5 this is no longer the case for pairs from the region (D).

Finally, we generalized the incompatibility of pairs of testers to incompatibility of arbitrary number of  $N$ -testers—that is, measurement setups that test networks of  $N$  causally related channels. The generalization is relatively straightforward and in this general settings we have related the robustness of incompatibility with the minimum-error discrimination of the associated normalizations. As it is known that incompatibility of observables and channels is rooted in many quantum information applications, we believe that the research program on incompatibility of testers we started in this paper, will have strong impact on future of information processing and also on the understanding of foundational questions about time and causal structure in quantum theory.

### Acknowledgments

We acknowledge support from the SRDA grant APVV-0808-12 (QETWORK), VEGA Grant No. 2/0125/13 (QUICOST), from the the Foundational Questions Institute (FQXi-RFP3-1325), from the National Natural Science Foundation of China through Grants 11450110096

and 11350110207, and from the 1000 Youth Fellowship Program of China. DR was supported via SASPRO Program No. 0055/01/01 (QWIN) and MS acknowledges support by the European Social Fund and the state budget of the Czech Republic under Operational Program Education for Competitiveness (Project No. CZ.1.07/2.3.00/30.0004) and by the Development Project of Faculty of Science, Palacky University

### Appendix A: Diagonal form of operator $D$

Let us first explicitly write operator  $D$

$$\begin{aligned} D = \frac{1-\lambda}{2} &\left[ (P_{-\varphi/2} - P_{\pi-\varphi/2}) \otimes P_{-\theta/2} \right. \\ &\left. + (P_{\varphi/2} - P_{\varphi/2-\pi}) \otimes P_{\theta/2} \right] \\ &+ \frac{\lambda\delta}{2} \left[ (P_{(\pi-\varphi)/2} - P_{-(\pi+\varphi)/2}) \otimes P_{-\pi/2} \right. \\ &\left. + (P_{(\varphi-\pi)/2} - P_{(\pi+\varphi)/2}) \otimes P_{\pi/2} \right], \quad (\text{A1}) \end{aligned}$$

where we used (18), (23). It can be written more compactly as

$$D = \frac{1-\lambda}{2}(H + THT) + \frac{\lambda\delta}{2}(K + TKT), \quad (\text{A2})$$

where

$$\begin{aligned} H &= (P_{-\varphi/2} - P_{\pi-\varphi/2}) \otimes P_{-\theta/2} \\ K &= (P_{(\pi-\varphi)/2} - P_{-(\pi+\varphi)/2}) \otimes P_{-\pi/2} \end{aligned} \quad (\text{A3})$$

and  $T \equiv \sigma_Z \otimes \sigma_Z$  is a tensor product of Pauli matrices.

Our aim is to show that operator  $D$  does not mix subspaces defined by projectors  $Q, Q^\perp$ , i.e.

$$QDQ^\perp = Q^\perp DQ = 0. \quad (\text{A4})$$

Thanks to hermicity of operator  $D$  it suffices to show  $QDQ^\perp = 0$ . We observe that  $TQT = Q$  and consequently  $[Q, T] = 0$ . Similarly,  $TQ^\perp T = Q^\perp$  implies  $[Q^\perp, T] = 0$ . This means it is crucial to calculate operators  $QHQ^\perp, QKQ^\perp$  and the remaining terms of  $QDQ^\perp$  can be obtained by conjugation with  $T$ . For such calculation the following formula is useful

$$P_\alpha P_\beta P_\gamma = |\alpha\rangle\langle\gamma| \cos \frac{\alpha-\beta}{2} \cos \frac{\beta-\gamma}{2}. \quad (\text{A5})$$

After a longer, but straightforward calculation one obtains

$$\begin{aligned} QHQ^\perp &= \frac{\sin \frac{\varphi}{2} \cos \frac{\theta}{2}}{2} \left[ \left| -\frac{\varphi}{2} \right\rangle \left\langle \frac{\varphi}{2} - \pi \right| \otimes \left| -\frac{\pi}{2} \right\rangle \left\langle \frac{\pi}{2} \right| \right. \\ &\quad \left. - \left| \frac{\varphi}{2} \right\rangle \left\langle \pi - \frac{\varphi}{2} \right| \otimes \left| \frac{\pi}{2} \right\rangle \left\langle -\frac{\pi}{2} \right| \right] \\ &\quad + \sin \varphi \frac{1 - \sin \frac{\theta}{2}}{2} \left| \frac{\varphi}{2} \right\rangle \left\langle \frac{\varphi}{2} - \pi \right| \otimes P_{\pi/2} \\ QKQ^\perp &= \left| -\frac{\varphi}{2} \right\rangle \left\langle \pi - \frac{\varphi}{2} \right| \otimes P_{-\pi/2} \end{aligned} \quad (\text{A6})$$

Thanks to Eq.(A6) it is easy to evaluate

$$\begin{aligned} QHQ^\perp + TQHQ^\perp T &= \sin \varphi \frac{1 - \sin \frac{\theta}{2}}{2} \times \\ &\times \left[ \left| \frac{\varphi}{2} \right\rangle \left\langle \frac{\varphi}{2} - \pi \right| \otimes P_{\pi/2} + \left| -\frac{\varphi}{2} \right\rangle \left\langle \pi - \frac{\varphi}{2} \right| \otimes P_{-\pi/2} \right] \\ QKQ^\perp + TQKQ^\perp T &= \\ &= \left[ \left| \frac{\varphi}{2} \right\rangle \left\langle \frac{\varphi}{2} - \pi \right| \otimes P_{\pi/2} + \left| -\frac{\varphi}{2} \right\rangle \left\langle \pi - \frac{\varphi}{2} \right| \otimes P_{-\pi/2} \right] \end{aligned} \quad (\text{A7})$$

where the terms with  $\cos \frac{\theta}{2}$  effectively disappeared due to conjugation. Finally, using Eqs. (A2), (A7) and definitions (15),(17) we get  $QDQ^\perp = 0$ , because

$$\frac{1 - \lambda}{2} \sin \varphi \frac{1 - \sin \frac{\theta}{2}}{2} + \frac{\lambda \delta}{2} = 0. \quad (\text{A8})$$

This allows us to write

$$D = (Q + Q^\perp)D(Q + Q^\perp) = QDQ + Q^\perp DQ^\perp. \quad (\text{A9})$$

Let us calculate  $QDQ$ . Direct calculation using (A5) shows that

$$\begin{aligned} QKQ &= \left[ P_{-\varphi/2} (P_{(\pi-\varphi)/2} - P_{-(\pi+\varphi)/2}) P_{-\varphi/2} \right] \otimes P_{-\pi/2} \\ &= \left[ \frac{1}{2} P_{-\varphi/2} - \frac{1}{2} P_{-\varphi/2} \right] \otimes P_{-\pi/2} = 0 \end{aligned} \quad (\text{A10})$$

and

$$\begin{aligned} QHQ &= \frac{1 - \sin \frac{\theta}{2}}{2} \cos \varphi |v_1\rangle \langle v_1| + \frac{1 + \sin \frac{\theta}{2}}{2} |v_2\rangle \langle v_2| \\ &\quad + \frac{1}{2} \cos \frac{\theta}{2} \cos \frac{\varphi}{2} (|v_1\rangle \langle v_2| + |v_2\rangle \langle v_1|) \end{aligned} \quad (\text{A11})$$

Thanks to Eqs. (A2),(A10),(A11) and the fact that  $T|v_1\rangle = |v_2\rangle$  we obtain

$$QDQ = \frac{1 - \lambda}{2} (QHQ + TQHQ^\perp T) = G. \quad (\text{A12})$$

The last thing we have to show is  $Q^\perp DQ^\perp = -SGS$ . Let us note the following identities

$$\begin{aligned} SHS &= -THT & ST &= -TS \\ SKS &= -TKT & SQS &= Q^\perp, \end{aligned} \quad (\text{A13})$$

which from Eqs. (A2), (A12) imply  $SDS = -D$  and

$$Q^\perp DQ^\perp = SQSDSQS = -SQDQS = -SGS. \quad (\text{A14})$$

Combining equations (A9), (A12), (A14) we obtain Eq. (27) we wanted to prove.

## Appendix B: Proof of Proposition 11

Since  $E_2 = I - E_1$  and  $F_2 = I - F_1$ , we can parameterize both POVMs by spectral decompositions of the effects  $E_1, F_1$

$$\begin{aligned} E_1 &= e_1 |u_1\rangle \langle u_1| + e_2 |u_2\rangle \langle u_2| \\ F_1 &= f_1 |w_1\rangle \langle w_1| + f_2 |w_2\rangle \langle w_2|, \end{aligned} \quad (\text{B1})$$

where  $e_i, f_j \in [0, 1]$  and  $\{|u_1\rangle, |u_2\rangle\}, \{|w_1\rangle, |w_2\rangle\}$  are two orthonormal qubit bases. Effects  $E_1, F_1$  as well as the corresponding POVMs can be convexly decomposed into four projective measurements (extremal POVMs),

$$E_1 = \sum_{a=1}^4 c_a E_1^a, \quad F_1 = \sum_{b=1}^4 d_b F_1^b, \quad (\text{B2})$$

where

$$\begin{aligned} E_1^1 &= 0, & F_1^1 &= 0, \\ E_1^2 &= |u_1\rangle \langle u_1|, & F_1^2 &= |w_1\rangle \langle w_1|, \\ E_1^3 &= |u_2\rangle \langle u_2|, & F_1^3 &= |w_2\rangle \langle w_2|, \\ E_1^4 &= I, & F_1^4 &= I. \end{aligned} \quad (\text{B3})$$

The decomposition in Eq. (B2) is unique and such that

$$\sum_{a=1}^4 c_a = 1, \quad \sum_{b=1}^4 d_b = 1. \quad (\text{B4})$$

The two outcome POVMs defined by effects  $E_1^1, E_1^4, F_1^1, F_1^4$  are trivial, i.e. their outcomes can be generated without actually measuring the quantum state. The first pair and the second pair of POVMs defined by effects  $E_1^2, E_1^3, F_1^2, F_1^3$  are related by relabeling of outcomes (e.g.  $E_1^2 = E_1^3, E_1^3 = E_1^2$ ).

We define 1-testers

$$\begin{aligned} \mathcal{A}^a &= \{A_1^a, A_2^a\}, & \mathcal{B}^b &= \{B_1^b, B_2^b\}, \\ A_k^a &= E_k^a \otimes P_{-\theta/2}, & B_k^b &= F_k^b \otimes P_{\theta/2}, \end{aligned}$$

where  $k = 1, 2$  and  $a, b = 1, 2, 3, 4$ . Showing that all pairs of testers  $\mathcal{A}^a$  and  $\mathcal{B}^b$  (for all  $a, b$ ) are  $\lambda$ -compatible will be later used to show compatibility of 1-testers  $\mathcal{A}$  and  $\mathcal{B}$ .

Firstly, the outcomes of trivial POVMs can be generated without measuring the state, and so it is clear that those 1-testers defined above that contain trivial POVM will be  $\lambda$ -compatible with  $\lambda$  obeying Proposition 4 with any other product 1-tester  $\mathcal{B} = \{F_1 \otimes \xi, F_2 \otimes \xi\}$ . For example, for  $A_1^4 = I \otimes \varrho$  it would suffice to choose  $\tilde{A}_1^4 = I \otimes \tilde{\varrho}, \tilde{B}_1 = F_1 \otimes \tilde{\xi}$  and

$$\begin{aligned} G_{11} &= (1 - \lambda)B_1 + \lambda\tilde{B}_1, & G_{12} &= (1 - \lambda)B_2 + \lambda\tilde{B}_2, \\ G_{21} &= 0, & G_{22} &= 0. \end{aligned}$$

Clearly,  $G_{11} + G_{21} = \tilde{B}_1, G_{12} + G_{22} = \tilde{B}_2$  and

$$\begin{aligned} G_{21} + G_{22} &= 0 = \tilde{A}_1^4, \\ G_{11} + G_{12} &= (1 - \lambda)I \otimes \xi + \lambda I \otimes \tilde{\xi} \\ &= (1 - \lambda)I \otimes \varrho + \lambda I \otimes \tilde{\varrho} = \tilde{A}_1^4. \end{aligned}$$

Thus, for  $\theta \geq 2 \arcsin(1/3) \approx 0.6797$  1-testers  $\mathcal{A}^a, \mathcal{B}^b$  are  $\lambda$ -compatible, because either one of them contains trivial POVM or the pair is unitarily equivalent to 1-testers in Proposition 10.

Now it suffices to show that  $\lambda$ -compatibility of 1-testers  $\mathcal{A}^a, \mathcal{B}^b$  for all  $a, b$  implies  $\lambda$ -compatibility of 1-testers  $\mathcal{A}, \mathcal{B}$  from the proposition 11. This can be done as follows. The fact that for  $\theta \geq 2 \arcsin(1/3)$  1-testers  $\mathcal{A}^a, \mathcal{B}^b$  are  $\lambda$ -compatible can be expressed using Proposition 2 by existence of operators  $G^{ab}$  satisfying

$$0 \leq G^{ab} \leq \bar{A}_1^a, \bar{B}_1^b, \quad (\text{B5})$$

$$\bar{A}_1^a + \bar{B}_1^b \leq G^{ab} + I \otimes \omega. \quad (\text{B6})$$

Let us note that  $\forall a, b$   $A_1^a + A_2^a = I \otimes P_{-\theta/2}$ ,  $B_1^b + B_2^b = I \otimes P_{\theta/2}$  and since  $\lambda$  is given by the lower bound (15) also  $\forall a, b$   $\tilde{A}_1^a + \tilde{A}_2^a = I \otimes P_{\pi/2}$ ,  $B_1^b + B_2^b = I \otimes P_{-\pi/2}$  and as a consequence the normalization of the joint tester  $\omega$  is the same  $\forall a, b$ .

To prove  $\lambda$ -compatibility of  $\mathcal{A}, \mathcal{B}$  we define the admixed 1-testers  $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$  and the joint 1-tester  $\mathcal{G}$

$$\begin{aligned} \tilde{A}_1 &= \sum_{a=1}^4 c_a \tilde{A}_1^a, & \tilde{B}_1 &= \sum_{b=1}^4 d_b \tilde{B}_1^b, \\ G &= \sum_{a,b=1}^4 c_a d_b G^{ab}. \end{aligned}$$

Let us remind that  $A_1 = \sum_{a=1}^4 c_a A_1^a$ ,  $B_1 = \sum_{b=1}^4 d_b B_1^b$ , which implies

$$\bar{A}_1 = \sum_{a=1}^4 c_a \bar{A}_1^a, \quad \bar{B}_1 = \sum_{b=1}^4 d_b \bar{B}_1^b.$$

Due to  $G^{ab} \geq 0$ ,  $c_a, d_b \geq 0$  we conclude  $G \geq 0$ , because  $G$  is a nonnegative sum of positive-semidefinite operators. We also easily get

$$\begin{aligned} G &= \sum_{a,b=1}^4 c_a d_b G^{ab} \leq \sum_{a,b=1}^4 c_a d_b \bar{A}_1^a = \bar{A}_1, \\ G &= \sum_{a,b=1}^4 c_a d_b G^{ab} \leq \sum_{a,b=1}^4 c_a d_b \bar{B}_1^b = \bar{B}_1, \end{aligned}$$

where we used (B5) and (B4).

Finally, we use Eq. (B5) to write

$$\bar{A}_1 + \bar{B}_1 = \sum_{a,b=1}^4 c_a d_b (\bar{A}_1^a + \bar{B}_1^b) \leq I \otimes \omega + G,$$

which concludes the proof.

### Appendix C: SDP for $\lambda$ -compatibility

Proposition 2 can be used to construct SDP for solving  $\lambda$ -compatibility of two two-outcome testers  $\mathcal{A} = \{A_1, A_2\}$

and  $\mathcal{B} = \{B_1, B_2\}$  such that  $A_1 + A_2 = I \otimes \varrho$  and  $B_1 + B_2 = I \otimes \xi$ . According to Def. 2 in  $\lambda$ -compatibility we search for the smallest  $\lambda$  such that the testers  $(1-\lambda)\mathcal{A} + \lambda\tilde{\mathcal{A}}$  and  $(1-\lambda)\mathcal{B} + \lambda\tilde{\mathcal{B}}$  are compatible for some  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{B}}$ . First of all, the necessary condition  $I \otimes \bar{\varrho} = I \otimes \bar{\xi}$  from Eq. (5) for the normalizations  $\bar{\varrho} = (1-\lambda)\varrho + \lambda\tilde{\varrho}$  and  $\bar{\xi} = (1-\lambda)\xi + \lambda\tilde{\xi}$  needs to be satisfied. Thus, in addition to search over operators  $G$ , we have to expand the search also over the mixed-in elements  $\tilde{A}_1, \tilde{B}_1$  and their normalizations  $\tilde{\varrho}, \tilde{\xi}$ . Defining

$$\begin{aligned} \bar{A}_i &= (1-\lambda)A_i + \lambda\tilde{A}_i, \\ \bar{B}_j &= (1-\lambda)B_j + \lambda\tilde{B}_j, \\ \omega &= (1-\lambda)\varrho + \lambda\tilde{\varrho}, \\ &= (1-\lambda)\xi + \lambda\tilde{\xi}, \end{aligned}$$

the problem can be recast as the following bilinear SDP

$$\begin{aligned} \text{Find} \quad & \inf \lambda \\ \text{subject to} \quad & 0 \leq \tilde{A}_1, \tilde{B}_1, \tilde{\varrho}, \tilde{\xi}, G \\ & G \leq \bar{A}_1, \bar{B}_1 \\ & \bar{A}_1 + \bar{B}_1 \leq G + I \otimes \omega, \\ & (1-\lambda)(\varrho - \xi) = \lambda(\tilde{\xi} - \tilde{\varrho}), \\ & \tilde{A}_1 \leq I \otimes \tilde{\varrho}, \tilde{B}_1 \leq I \otimes \tilde{\xi}, \\ & \text{tr}[\tilde{\varrho}] = \text{tr}[\tilde{\xi}] = 1, \end{aligned}$$

where the last condition comes from the common normalization to  $\omega$ . We can linearize the program by rescaling relevant operators by  $1/\lambda$ . Using the definition of  $\omega$  and setting  $\mu = (1-\lambda)/\lambda$ ,  $H = \frac{1}{\lambda}G$  the SDP program can be equivalently stated as

$$\begin{aligned} \text{Find} \quad & \mu_0 := \sup \mu \\ \text{subject to} \quad & 0 \leq \tilde{A}_1, \tilde{B}_1, \tilde{\varrho}, \tilde{\xi}, H, \\ & H \leq \mu A_1 + \tilde{A}_1, \\ & H \leq \mu B_1 + \tilde{B}_1, \\ & \mu(A_1 + B_1 - I \otimes \xi) + \tilde{A}_1 + \tilde{B}_1 \leq H + I \otimes \tilde{\xi}, \\ & \mu(\varrho - \xi) = \tilde{\xi} - \tilde{\varrho}, \\ & \tilde{A}_1 \leq I \otimes \tilde{\varrho}, \tilde{B}_1 \leq I \otimes \tilde{\xi}, \\ & \text{tr}[\tilde{\varrho}] = \text{tr}[\tilde{\xi}] = 1, \end{aligned}$$

where the unknown objects are  $\mu, H, \tilde{A}_1, \tilde{B}_1, \tilde{\varrho}, \tilde{\xi}$ . Then the minimal  $\lambda$  is determined as

$$\lambda_0 = \frac{1}{1 + \mu_0}.$$

- 
- [1] E. H. Kennard, Zur Quantenmechanik einfacher Bewegungstypen, *Zeitschrift für Physik* **44**, 326 (1927).
- [2] H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel (Leipzig, 1928).
- [3] H.P. Robertson, The Uncertainty Principle, *Phys. Rev.* **34**, 163–164 (1929).
- [4] E. Schrödinger, Zum Heisenbergschen Unschärfeprinzip, *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse* **14**, 296–303 (1930).
- [5] B.-G. Englert, Fringe Visibility and Which-Way Information: An Inequality, *Phys. Rev. Lett.* **77**, 2154 (1996).
- [6] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeitschrift für Physik* **43**, 172–198 (1927).
- [7] M.M. Wolf, D. Perez-Garcia, C. Fernandez, Measurements incompatible in Quantum Theory cannot be measured jointly in any other local theory, *Phys. Rev. Lett.* **103**, 230402 (2009).
- [8] R. Uola, T. Moroder, O. Gühne, Joint measurability of generalized measurements implies classicality, *Phys. Rev. Lett.* **113**, 160403 (2014).
- [9] V. Scarani, S. Iblisdir, N. Gisin, A. Acin, Quantum cloning, *Rev. Mod. Phys.* **77**, 1225–1256 (2005).
- [10] P. Busch, “No Information Without Disturbance”: Quantum Limitations of Measurement, chapter in *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle*, eds. W.C. Myrvold; J. Christian, pages 229–256 (Springer, Dordrecht, 2009).
- [11] T. Heinosaari, J. Kiukas, D. Reitzner, Noise robustness of the incompatibility of quantum measurements, *Phys. Rev. A* **92**, 022115 (2015).
- [12] P. Busch, Unsharp reality and joint measurements for spin observables, *Phys. Rev. D* **33**, 2253–2261 (1986).
- [13] P. Lahti, Coexistence and joint measurability in quantum mechanics, *Int. J. Theor. Phys.* **42**, 893–906 (2003).
- [14] P. Lahti, S. Pulmannová, Coexistent observables and effects in quantum mechanics, *Rep. Math. Phys.* **39**, 339–351 (1997).
- [15] P. Stano, D. Reitzner, T. Heinosaari, Coexistence of qubit effects, *Phys. Rev. A* **78**, 012315 (2008).
- [16] P. Busch, H.-J. Schmidt, Coexistence of qubit effects, *Quant. Info. Proc.* **9**, 143 (2010).
- [17] Yu S., Liu N.-L., Li L., Oh C.H., Joint measurement of two unsharp observables of a qubit, *Phys. Rev. A* **81**, 062116 (2010).
- [18] T. Heinosaari, J. Kiukas, and D. Reitzner, Coexistence of effects from an algebra of two projections, *J. Phys. A* **47**, 225301 (2014).
- [19] R. Kunjwal, C. Heunen, T. Fritz, All joint measurability structures are quantum realizable, *Phys. Rev. A* **89**, 052126 (2014).
- [20] G. Ludwig, *Foundations of Quantum Mechanics I* (Berlin: Springer, 1983).
- [21] T. Heinosaari, T. Miyadera, D. Reitzner, Strongly Incompatible Quantum Devices, *Foundations of Physics* **44**, 34–57 (2014).
- [22] G. Chiribella, G. M. D’Ariano, P. Perinotti, Memory Effects in Quantum Channel Discrimination, *Phys. Rev. Lett.* **101**, 180501 (2008).
- [23] T. Heinosaari, M. Ziman, *The mathematical language of quantum theory: From uncertainty to entanglement* (Cambridge University Press, Cambridge, UK, 2012).
- [24] G. Chiribella, G. M. D’Ariano, P. Perinotti, Theoretical framework for quantum networks, *Phys. Rev. A* **80**, 022339 (2009).
- [25] M. Ziman, Process POVM: A mathematical framework for the description of process tomography experiments, *Phys. Rev. A* **77**, 062112 (2008).
- [26] P. Busch, T. Heinosaari, J. Schultz, N. Stevens, Comparing the degrees of incompatibility inherent in probabilistic physical theories, *Europhysics Letters* **103**, 10002 (2013).
- [27] E. Haapasalo, Robustness of incompatibility for quantum devices, *J. Phys. A* **48**, 255303 (2015).
- [28] T. Heinosaari, A simple sufficient condition for the coexistence of quantum effects, *J. Phys. A: Math. Theor.* **46**, 152002 (2013).
- [29] G. Chiribella, G. M. D’Ariano, P. Perinotti, Quantum Circuits Architecture, *Physical Review Letters* **101**, 180501 (2008).
- [30] G. Gutoski, J. Watrous, Towards a general theory of quantum games, in *Proceedings of the thirty-ninth annual ACM Symposium on Theory of computing*, pp. 565–574 (2007).
- [31] G. Chiribella, Optimal networks for quantum metrology: semidefinite programs and product rules, *New Journal of Physics* **14**, 125008 (2012).
- [32] G. Gutoski, On a measure of distance for quantum strategies, *J. Math. Phys.* **53**, 032202 (2012).
- [33] T. Heinosaari, J. Schultz, A. Toigo, M. Ziman, Maximally incompatible quantum observables, *Phys. Lett. A* **378**, 1695–1699 (2014).